ODE. Tuesday Sept. 27. Integrating Factor & Separable ODE.

Separable ODE.

If an ODE can be written in the form

$$h(y)\frac{dy}{dx} = g(x),$$

for some functions $h(y)$ and $g(x)$,

then it is called separable and can be

solved by integrating both sides w.r.t. x:

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

so

$$\int h(y)\,dy = \int g(x)\,dx.$$

ANN.

1. Demos this W: 3p.m. B207 thereafter B312.
2. Add'l writing prompt for notebooks this week - rel. to modelling. worth 3/10 notebk pts. upld to canvas
3. Meet me today, tmrw, R discuss topic

Example 1.

$$x \frac{dy}{dx} = 2(y-4)$$

If $x \neq 0$, $y \neq 4$ then:

$$\frac{1}{y-4}\frac{dy}{dx} = \frac{2}{x}$$

$$\int \frac{1}{y-4}\,dy = \int \frac{2}{x}\,dx$$

$$\ln|y-4| = 2\ln|x| + c$$

$$y-4 = e^{2\ln|x|+c} = Ae^{2\ln|x|} = A\left(e^{\ln|x|}\right)^2 = Ax^2$$

$$y = Ax^2 + 4$$

Check: $\frac{dy}{dx} = 2xA$, so $x\frac{dy}{dx} = 2x^2A \stackrel{?}{=} \underbrace{2(y-4)}$

$$= 2(Ax^2+4-4)$$

✓ $= 2Ax^2$

what if $\left.\begin{matrix} y=4 \\ x=0 \end{matrix}\right\}$? same. No problem/discontinuities.

Just be aware that our soln method may have introduced them

Separable ODE.

Ex. 2
IVP

$$\frac{dy}{dx} = ye^x, \quad y(0) = 2e.$$

If $y \neq 0$ then $\frac{1}{y}\frac{dy}{dx} = e^x$

$$\int \frac{1}{y}\, dy = \int e^x\, dx$$

$$\ln|y| = e^x + c$$

$$y = \exp(e^x + c)$$
$$= A\, e^{(e^x)}$$

Then

$$y(0) = A e^{(e^0)} = Ae^1 = Ae,$$

and by the initial condition,

$y(0) = 2e.$

Therefore, $A = 2$ and $y(x) = 2e^{(e^x)}$

$\cancel{e^{(e^x)} \neq (e^e)^x = e^{ex}}$

$$\boxed{\begin{aligned}(a^b)^c &= \overbrace{(a^b)(a^b)\cdots(a^b)}^{c \text{ times}} = \overbrace{aa\cdots a}^{bc \text{ times}} \\ a^{(b^c)} &= \underbrace{a\cdot a\cdot a\cdots a}_{b^c \text{ times}} \qquad bc \neq b^c\end{aligned}}$$

Integrating Factors.

Def.

A linear first-order eq'n is of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

~~(Does this mesh with Tenenbaum + Pollard?)~~ ~~LoA~~

(Check this with what we knew abt. classifying ODE.)

~~a linear eq'n is like y = mx + b~~

~~Wouldn't it be nice if we had~~ ~~could easily solve~~

Recall product rule: $\frac{d}{dx}\left[g(x)\,y\right] = g(x)\frac{dy}{dx} + g'(x)\,y.$

Nice to have $\frac{d}{dx}\left[h(x)\,y\right] = Q(x).$ (Int. both sides, divide by $h(x)$.)

Why don't we try to find some function $\rho(x)$ to mult. by?

would have

$$\frac{d}{dx}\left[\rho(x)\,y\right] = \cancel{\rho(x)\frac{dy}{dx}} + \rho'(x)\,y \quad := \cancel{\rho(x)\frac{dy}{dx}} + \rho(x)P(x)\,y$$

so $\rho'(x) = \rho(x)\,P(x)$

or
$$\frac{d\rho}{dx} = \rho P(x)$$

$$\frac{1}{\rho}\frac{d\rho}{dx} = P(x)$$

$$\int \frac{1}{\rho}\, d\rho = \int P(x)\, dx$$

$$\ln|\rho| = \int P(x)\, dx$$

$$\rho = A \exp\left(\int P(x)\, dx\right).$$

we just need any ρ that works, so let $A := 1$.

Then
$$\cancel{\rho(x)\frac{dy}{dx} + \rho(x)P(x)y = \rho(x)Q(x)}$$

~~and~~ by our design, $\rho(x)$ is s.t. the LHS:

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)y = \rho(x)\frac{dy}{dx} + \rho'(x)y = \frac{d}{dx}[\rho y]$$

and so we mult. both sides of ODE:

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)y = \rho(x)Q(x)$$

$$\frac{d}{dx}\left[\rho(x)y(x)\right] = \rho(x)Q(x)$$

So
$$\rho(x)y = \int \rho(x)Q(x)\, dx$$

$$\Rightarrow \quad y = \frac{1}{\rho(x)}\int \rho(x)Q(x)\, dx.$$

Int. Factor.

① Compute $\rho(x) := e^{\int P(x)\,dx}$, taking $+C = 0$

② Mult. both sides of ODE by $\rho(x)$.

③ Obtain $\frac{d}{dx}\left[\rho(x)\, y\right] = \rho(x)\, Q(x)$

④ Integrate: $\rho(x)\, y = \int \rho(x)\, Q(x)\, dx \leftarrow \underline{\underline{+C}}$

(matters here)

⑤ Solve for y.

Example 3

$$\frac{dy}{dx} - y = \frac{11}{8} e^{-x/3}, \quad y(0) = -1.$$

$P(x) = -1$, $Q(x) = \frac{11}{8} e^{-x/3}$

① $\rho(x) := e^{\int P(x)\,dx} = \exp\left(\int -1\, dx\right) = \exp(-x) = e^{-x}$.

② $e^{-x}\frac{dy}{dx} - e^{-x} y = \frac{11}{8} e^{-x/3}\left(e^{-x}\right) = \frac{11}{8} e^{-4x/3}$

③ $\frac{d}{dx}\left[e^{-x} y\right] = \frac{11}{8} e^{-4x/3}$

④ $e^{-x} y = \int \frac{11}{8} e^{-4x/3}\, dx = \frac{11}{8}\left(-\frac{3}{4}\right) e^{-4x/3} + C$

⑤ $y = -\frac{11}{8}\left(\frac{3}{4}\right) e^{-4x/3}\left(e^{x}\right) + Ce^{x} = -\frac{33}{32} e^{-x/3} + Ce^{x}$.

I.C.: $y(0) = -33/32 + C = -1$ when $C = 1/32$. So $y(x) = \frac{1}{32} e^{x} - \frac{33}{32} e^{-x/3}$

Int. Factor.

Example 4. $y' = 1 + x + y + xy, \qquad y(0) = 0.$

$$\frac{dy}{dx} - y - xy = 1 + x$$

$$\frac{dy}{dx} - (1+x)y = 1 + x$$

Linear 1st ord.

(1) $g(x) := \exp\left(\int -(1+x)\,dx\right)$

$$= \exp\left(-x - \tfrac{1}{2}x^2\right)$$

(2) $\exp\left(-x-\tfrac{1}{2}x^2\right)\frac{dy}{dx} - (1+x)\exp\left(-x-\tfrac{1}{2}x^2\right)y = (1+x)\exp\left(-x-\tfrac{1}{2}x^2\right)$

(3) $\frac{d}{dx}\left[\exp\left(-x-\tfrac{1}{2}x^2\right)y\right] = (1+x)\exp\left(-x-\tfrac{1}{2}x^2\right)$

(4) $\exp\left(-x-\tfrac{1}{2}x^2\right)y = \int (1+x)\exp\left(-x-\tfrac{1}{2}x^2\right)dx$

$$= -\exp\left(-x-\tfrac{1}{2}x^2\right) + C$$

(5) $y = -\exp\left(-x-\tfrac{1}{2}x^2\right)\exp\left(x+\tfrac{1}{2}x^2\right) + C\exp\left(x+\tfrac{1}{2}x^2\right)$

$$= C\exp\left(\tfrac{1}{2}x^2+x\right) - 1$$

$y(0)=0$ implies $C - 1 = 0 \Rightarrow C = 1.$

$y = \exp\left(\tfrac{1}{2}x^2+x\right) - 1. \qquad y' = (1+x)e^{\frac{1}{2}x^2+x}$

Check: $(1+x)e^{\frac{1}{2}x^2+x} \overset{?}{=} \cancel{1} + \exp\left(\tfrac{1}{2}x^2+x\right) \cancel{-1} + \cancel{x} + x\exp\left(\tfrac{1}{2}x^2+x\right) \cancel{-x}$
$= (1+x)\exp(\cdot)$ ✓