Review so far & Demo practice. Oct. 4, 2016.

Can do 1st 8 demos:

1. Classification
2. Implicit & explicit sol'ns
3. Multiplicity of sol'ns & initial cond'ns
4. Exact ODE
5. Separation of variables
6. Integrating factor
7. First-order linear ODE
8. Bernoulli equations.

We'll spend 7-8 min. on each topic.

Classification.

- Ordinary / partial

- Order

- Linearity.

Looks like: $f_0(x) + f_1(x)\,y + f_2(x)\dfrac{d^2y}{dx^2} + f_3(x)\dfrac{d^3y}{dx^3} + \dots + f_n(x)\dfrac{d^ny}{dx^n} = 0$

Idea: have no terms like $\left[d^2y/dx\right]^2$ or $\left[y'''\right]y$, etc:

Examples.

① $$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2}$$

② $$\frac{dy}{dx} = y^2 \quad \text{or} \quad dy = y^2\,dx$$

③ $$y' + y = e^x \quad \text{or} \quad dy + (y - e^x)\,dx = 0$$

④ $$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} = 3\cos(2t)$$

Implicit & Explicit.

Verify a sol'n.

Explicit

(1) $x^2y'' + xy' - y = \ln(x)$.

Verify $y_1 = x - \ln(x)$, $y_2 = \frac{1}{x} - \ln(x)$.

$$\frac{dy_1}{dx} = 1 - \frac{1}{x}, \quad \frac{d^2y_1}{dx^2} = \frac{1}{x^2}$$

$$\frac{dy_2}{dx} = \frac{-1}{x^2} - \frac{1}{x}, \quad \frac{d^2y_2}{dx^2} = \frac{2}{x^3} + \frac{1}{x^2}$$

$$x^2y_1'' + xy_1' - y_1 = x^2\left(\frac{1}{x^2}\right) + x\left(1 - \frac{1}{x}\right) - (x - \ln x)$$

$$= 1 + x - 1 - x + \ln x = \ln x \checkmark$$

$$x^2y_2'' + xy_2' - y_2 = x^2\left(\frac{2}{x^3} + \frac{1}{x^2}\right) + x\left(-\frac{1}{x^2} - \frac{1}{x}\right) - \left(\frac{1}{x} - \ln x\right)$$

$$= \frac{2}{x} + 1 - \frac{1}{x} - 1 - \frac{1}{x} + \ln x = \ln x \checkmark$$

(2) $x + y\frac{dy}{dx} = 0$. Verify $x^2 + y^2 = 4$.

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[4]$$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

Then $x + y\frac{dy}{dx} = x + y\left(\frac{-x}{y}\right) = x - x = 0$ ✓.

Must be sure $y \neq 0$, i.e., $x \neq \pm 2$.

Verifying.

(3) $y^2-1-(2y+xy)y'=0$, $y^2-1=(x+2)^2$

(4) $y''-y=0$, $y=ae^x+be^{-x}$.

Multiplicity & i.c.

(1) $y=2x+3+ae^x+be^{2x}$

is a ___-parameter family of sol'ns of $y''-3y'+2y-4x=0$

(2) Find a & b if $y(0)=5$ and $y'(0)=3$.

(3) $x=16t^2+c_1t+c_2$ solves an ODE

$c_2=10$

$x(0)=10$

$c_1=20$

$x'(0)=20$

Separation of variables,

① $$\frac{dr}{d\theta} = -\sin\theta$$

$$\int \frac{dr}{d\theta}\, d\theta = \int -\sin\theta \, d\theta$$

$$\int dr = \int -\sin\theta \, d\theta$$

$$r = \cos\theta + c$$

② $$(y^2+1)\,dx - (x^2+1)\,dy = 0$$

$$\frac{1}{y^2+1}\frac{dy}{dx} = \frac{1}{x^2+1}$$

$$\int \frac{1}{y^2+1}\, dy = \int \frac{1}{x^2+1}\, dx$$

$$\arctan(y) = \arctan(x) + c$$ implicit

$$y = \tan(\arctan(x) + c)$$ explicit

③ $$y' = 6x\,(y-1)^{2/3}$$

$$y = 1 + (x^2+c)^3$$

$$\frac{1}{(y-1)^{2/3}}\frac{dy}{dx} = 6x$$

④ $$\frac{dy}{dx} + 1 = 2y$$

$$\frac{dy}{dx} = 2y - 1$$

$$\frac{1}{2y-1}\frac{dy}{dx} = 1$$

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Exact ODE

indivi exercise

① $y^3\,dx + 3xy^2\,dy = 0$

$$\begin{cases} M\,dx + N\,dy = 0 \\ F = \int M\,dx + g(y) \\ \dfrac{\partial F}{\partial y} = N \end{cases}$$

Find $y = kx^{-1/3}$

② $\cancel{x^2 y' = xy + y^2}$ / $(x-6y)y' = 4x - y$

do this tgthr

$\cancel{\underbrace{x^2}_{N}\,dy - \underbrace{(xy+y^2)}_{M}\,dx = 0}$

$\cancel{\text{want } \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

$dx\,(y-4x) + (x-6y)\,dy = 0$

Check: $\dfrac{\partial}{\partial y}(y-4x) \overset{?}{=} \dfrac{\partial}{\partial x}(x-6y)$

$$F := \int y - 4x\,dx + g(y) = xy - 2x^2 + g(y)$$

$$\frac{\partial F}{\partial y} = x + g'(y) := x - 6y, \text{ so } g'(y) = -6y$$

$$\int g'(y)\,dy = \int -6y\,dy$$

$$g(y) = -3y^2 + c$$

So $F(x,y) = xy - 2x^2 - 3y^2 + c = 0$ solves the ODE

check: $\dfrac{d}{dx}\left(xy - 2x^2 - 3y^2 + c\right) = \dfrac{d}{dx}(0) \Leftrightarrow y + x\dfrac{dy}{dx} - 4x - 6y\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{4x - y}{x - 6y} \quad ✓$$