ODE Add'l Notes: Solving Exact Differential Equations.

An <u>exact</u> diff'l eq'n is of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

or $$M(x,y)\,dx + N(x,y)\,dy = 0,$$

where $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To handle/solve exact ODEs:

1. Verify exactness by checking that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

2. Let $F(x,y) := \int M(x,y)\,dx + g(y)$, where $g(y)$ is an "arbitrary constant of integration" as far as x is concerned.

3. Solve for $g(y)$ by setting $\frac{\partial F}{\partial y} = N(x,y)$.

4. This yields an implicit sol'n in the form $F(x,y) = C$, for some constant C.

5. Verify that $F(x,y) = C$ solves the ODE (perhaps by direct substitution).

Example. Solve the ODE

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0.$$

Sol'n. Here, $M(x,y) = 6xy - y^3$ and $N(x,y) = 4y + 3x^2 - 3xy^2$.

① Notice that $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[6xy - y^3\right] = 6x - 3y^2$

and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[4y + 3x^2 - 3xy^2\right] = 6x - 3y^2$,

and indeed, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. So the ODE is exact.

② Let
$$\begin{aligned} F(x,y) &:= \int M(x,y)\,dx + g(y) \\ &= \int 6xy - y^3\,dx + g(y) \\ &= 3x^2y - y^3x - g(y). \end{aligned}$$

③ $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[3x^2y - y^3x - g(y)\right] = 3x^2 - 3y^2x - g'(y).$$

If $\dfrac{\partial F}{\partial y} = N(x,y)$, then $3x^2 - 3y^2x - g'(y) = 4y + 3x^2 - 3xy^2$

So $-g'(y) = 4y \Rightarrow g'(y) = -4y$.

If $g'(y) = -4y$, then $g(y) = \int -4y\,dy = 2y^2$. So $F(x,y) = 3x^2y - y^3x + 2y^2$

* Can drop "+c" for this 1-param. family!

④ $F(x,y) = 3x^2y - y^3x + 2y^2$, so $3x^2y - y^3x + 2y^2 = C$ solves the ODE.

Sol'n, ct'd.

(5) Verify that $3x^2y - y^3x + 2y^2 = C$ gives an implicit soln to $(6xy - y^3) + (4y + 3x^2 - 3xy^2)\frac{dy}{dx} = 0$.

Well, if $3x^2y - y^3x + 2y^2 = C$, then $6xy + 3x^2\frac{dy}{dx} - 3y^2x\frac{dy}{dx} - y^3 + 4y\frac{dy}{dx} = 0$

so $\frac{dy}{dx}\left[3x^2 - 3y^2x + 4y\right] = y^3 - 6xy$, and thus

$$\frac{dy}{dx} = \frac{y^3 - 6xy}{3x^2 - 3y^2x + 4y}.$$

Substituting this into the ~~left~~ hand side of the ODE,

$$
\begin{aligned}
(6xy - y^3) + (4y + 3x^2 - 3xy^2)\frac{dy}{dx} &= 6xy - y^3 + (4y + 3x^2 - 3xy^2)\left(\frac{y^3 - 6xy}{3x^2 - 3y^2x + 4y}\right) \\
&= 6xy - y^3 + (y^3 - 6xy) \\
&= 0,
\end{aligned}
$$

just as the right-hand side of the ODE requires.