Oct. 18: Lin. Indep. (194 $\frac{1}{2}$ B); Wronskian (163B).

Def. A set of functions $\{f_1(x), f_2(x), f_3(x), \ldots, f_n(x)\}$ is said t/b linearly <u>dependent</u> if $\exists c_1, c_2, \ldots, c_n \in \mathbb{R}$ that are <u>not</u> <u>all</u> zero, s.t.

linear combination
$$\left\{ c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \cdots + c_n f_n(x) = 0 \right.$$

for <u>every</u> $x \in I$. (We assume the functions f_i are all defined in I.)

If no such set of constants exists, then the set is said t/b linearly <u>independent</u>.

<u>Example</u> (1) $\{x, -2x, -3x, 4x\}$, I is $(-\infty, \infty)$

✓ x is defined on $(-\infty, \infty)$

✓ $-2x$ —— " ——

✓ $-3x$ —— " ——

✓ $4x$ —— " ——

Try to find c_1, c_2, c_3, and c_4 not <u>all</u> zero s.t.

$$c_1(x) + c_2(-2x) + c_3(-3x) + c_4(4x) = 0 \quad \forall x \in \underset{\underset{(-\infty,\infty)}{I}}{\mathbb{R}}$$

$$x[c_1 - 2c_2 - 3c_3 + 4c_4] = 0$$

$c_1 := 2$, $c_2 := 1$, $c_3 := c_4 := 0$ works, thus $\{x, -2x, -3x, 4x\}$ is lin. <u>DEP</u>.

Example ② $\{x^p, x^q\}$, $p \neq q$, $\left.\begin{array}{l} x>0 \text{ or } x<0 \\ \mathbb{R}\setminus\{0\} \end{array}\right\} I$

✓ x^p is defined on I

✓ x^q —— " ————

Either find $c_1, c_2 \in \mathbb{R}$ not both zero, s.t.

$$c_1 x^p + c_2 x^q = 0 \quad \forall x \in \mathbb{R}\setminus\{0\}.$$

For a contradiction, assume $\exists c_1, c_2 \in \mathbb{R}$, not both zero, s.t. $c_1 x^p + c_2 x^q = 0$. WLOG, let $c_1 \neq 0$.

Then since $c_1 x^p = -c_2 x^q$, we have $x^{p-q} = -\dfrac{c_2}{c_1}$.

(we could do the division because x doesn't assume the value $0 \notin I$, and bc. $c_1 \neq 0$.)

So $$\underbrace{x^{p-q}}_{\text{varies with } x} = \underbrace{-\frac{c_2}{c_1}}_{\text{does not vary (constant)}} \quad \forall x \in \mathbb{R}\setminus\{0\}.$$

Contradiction! # $\rightarrow\!\leftarrow$ ↯

Therefore, $\nexists c_1, c_2 \in \mathbb{R}$ not both zero s.t. $c_1 x^p + c_2 x^q = 0$; i.e., $\{x^p, x^q\}$ is lin. <u>IN</u>dep.

Determinants.

Def. (The determinant of a 2×2 matrix)

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ has determinant

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) =: \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - cb = ad - cb.$$

Example: $\det\left(\begin{bmatrix} 1 & 0 \\ 14 & 5 \end{bmatrix}\right) = 1\cdot 5 - 14\cdot 0 = 5$

Def'n. The <u>Wronskian</u> of $\{f_1(x), f_2(x), \ldots, f_n(x)\}$

is
$$\det\left(\begin{bmatrix} f_1^{(0)}(x) & f_2^{(0)}(x) & \cdots & f_n^{(0)}(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}\right)$$

Example. $\{x, e^x\}$ has Wronskian:

$$W(\{x, e^x\}) = \det\left(\begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix}\right) = xe^x - 1\cdot e^x$$
$$= e^x(x-1).$$

<u>Theorem.</u> Let $S := \{ f_1(x), f_2(x), \dots, f_n(x) \}$.

If S is lin. <u>dep.</u> on the interval I,

Then $W(S) \equiv 0$ on I.

(That is, $W(S)$ "is identically zero", i.e., $W(S) = 0$ $\forall x \in I$.)

Lin. dep. $\Rightarrow$ $W \equiv 0$

<u>Contrapositive</u>: If $W(S) \not\equiv 0$ on I,

Then S is linearly <u>in</u>dependent on I.

(i.e., if $\exists x \in I$ s.t. $W(S) \neq 0$, then S is l. <u>indep.</u> on I)

<u>Converse</u>: ~~$W \equiv 0 \Rightarrow$ lin. dep.~~ !

counterexample: $\{ x^2, x|x| \}$

Example. $\{x, e^x\}$, $I = \mathbb{R}$

✓ x is dif. on $\mathbb{R}$
✓ e^x —— " ——

$$W(\{x, e^x\}) = \det\left(\begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix}\right) = xe^x - 1 \cdot e^x = e^x(x-1).$$

Observe that for $x = \pi$, $W(\{x, e^x\}) = e^\pi(\pi - 1) \neq 0$,

and so $\{x, e^x\}$ is lin. indep.

Example. $\{x^2, x|x|\}$, $I := (-2, 2)$ ✓ x^2 dif on I
✓ $x|x|$ —— " ——

Observe: $\frac{d}{dx}[x|x|] = x\frac{d}{dx}(|x|) + |x|$

$$= x|x|' + |x|$$

So
$$W(\{x^2, x|x|\}) = \det\left(\begin{bmatrix} x^2 & x|x| \\ 2x & x|x|' + |x| \end{bmatrix}\right)$$
$$= x^2(x|x|' + |x|) - 2x \cdot x|x|$$
$$= x^3|x|' + x^2|x| - 2x^2|x| = \underline{\underline{x^3|x|' - x^2|x|}}.$$

Case I ($x > 0$): Then $|x|' = 1$, because $|x| = x$ for $x > 0$.

So $W(S) = x^3 \cdot 1 - x^2 \cdot x = x^3 - x^3 = 0 \quad \forall x > 0$.

Case II ($x < 0$): Then $|x| = -x$, so $|x|' = -1$.

So $W(S) = x^3(-1) - x^2(-x) = -x^3 + x^3 = 0 \quad \forall x < 0$

Case III ($x = 0$): Then $|x| = x = 0$, so $W(S) = x^2(x|x|' - |x|)$
$= 0$.

So $\forall x$, $W(S) = 0$. Thus, $W(S) \equiv 0$ on $(-2, 2)$.

However, consider the linear comb.

$$c_1 x^2 + c_2 x|x| = 0. \quad \text{with } c_1, c_2 \text{ not both } 0$$
(i.e., set is lin. dep.)

Since this l.c. is 0 $\forall x \in (-2, 2)$, then choose $x = 1$:

$$c_1 + c_2 = 0$$

Choose $x = -1$: $\quad c_1 - c_2 = 0$.

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0 \Rightarrow c_1 = c_2. \quad \text{So } c_1 + c_1 = 0$$
$$2c_1 = 0 \Rightarrow c_1 = 0$$
$$\Rightarrow c_2 = 0.$$

$\lightning$ linear

Contradiction! Our assumption of dependence was false.